Reduction of Controller Fragility by Pole Sensitivity Minimization

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Abstract—This note presents a method for the reduction of controller fragility. The method is based on the sensitivity of closed-loop poles to perturbations in the controller parameters. By means of a space state parameterization of the controller, the closed-loop pole sensitivity can be reduced. A controller fragility measure based on the closed-loop pole sensitivity is proposed. Conditions for the optimal state-space realization of the controller are presented, along with a numerical method for obtaining the solution.

Index Terms—Controller fragility, controller implementation, optimal structure, pole sensitivity, stability.

I. INTRODUCTION

The problem of "fragile" controllers has recently been raised by Keel and Bhattacharyya [1]: a controller is fragile in the sense that very small perturbations in the coefficients of the designed controller destabilize the closed-loop control system. This is an important and fundamental issue in control system design.

In the usual design process, the assumption is often made that the controller can be implemented exactly. This assumption is to some extent valid, since, clearly, the plant uncertainty is the most significant source of uncertainty in the control system, whilst controllers are implemented with high precision hardware. However, there will inevitably be some amount of uncertainty in the controller, a fact that is sometimes ignored in advanced robust control design. If the controller is implemented by analog means, there are some tolerances in the analog components. More commonly, the controller will be implemented digitally, and consequently there will be some rounding the controller parameters. The rounding effects are even more problematic if, for reasons of safety, cost and execution speed, the implementation is with fixed-point rather than floating point processors. The question of controller fragility has only recently been explicitly raised, but it is certainly not new (see [2] for a brief overview). In fact, the closely related problem of finite word-length implementations of controllers was considered over 30 years ago [3], and has been the subject of continuing work since [4]–[11]. It is clear that many of these developed techniques can be applied directly to the controller fragility problem.

A number of examples of fragile optimal controllers are presented by Keel and Bhattacharyya [1], where the controller has been designed to tolerate uncertainty in the plant, however, very small perturbations on the controller parameters cause the closed-loop system to go unstable. This is not a surprising result, because the controller parameter perturbations are on the coefficients of the controller implemented as the ratio of two polynomials, and such representations are known to be inherently ill-conditioned [12, p. 171]. In fact, as pointed out in [2], the controller fragility will depend upon the particular realization of the controller. The question then arises of what is the best controller realization so as to minimize the effect of controller parameter errors on the closed-loop system. One approach to the problem has been through the minimization of pole/eigenvalue sensitivities to controller parameter perturbations. This was first considered for the open-loop controller eigenvalues by [13], and subsequently by [14, pp. 127–158] who solved the problem for state space controller realizations using the weighted sum of the 2-norms of the open loop eigenvalue sensitivities. The case of the closed loop system eigenvalues for state space controller realizations has been considered by [15], [16] using the maximum of the 2-norms of the closed-loop eigenvalue sensitivities, but the optimization problem is not totally solved [17]. A similar, less conservative, approach has been proposed by [16] using the maximum of the 1-norms of the closed loop eigenvalue sensitivities, but again the optimization problem is not solved in general and requires nonlinear programming to find local solutions [18]. The approach taken in this paper is to use the weighted sum of the 2-norms of the closed loop eigenvalue sensitivities; this allows the subsequent optimization problem to be completely solved using existing techniques.

The paper is organized as follows. In Section II, a measure of the controller fragility based on the weighted sum of the 2-norms of the sensitivities of the closed-loop poles/eigenvalues to perturbations in the controller parameters is proposed. Both state-space parameterizations and (to enable comparison with the results of [1]) single-input single-output (SISO) transfer function parameterizations are considered. In Section III, the problem of minimizing the measure for controller state-space parameterizations is formulated. Conditions for the optimal state-space realization of the controller are presented, and a method of solution based on gradient flow methods [19] is developed. In Section IV, the examples presented by Keel and Bhattacharyya [1], are studied, and the fragility shown to be dramatically reduced. Some concluding remarks are made in Section V.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. System Realizations and Controller Parameterizations

Consider the general feedback control system shown in Fig. 1. Let the plant be $G(\theta)$, where $\theta$ is a generalized operator (it could be the Laplace operator, $s$, for continuous systems, or the $z$ or $\delta$ operator for discrete systems), and let the controller be $H(\theta, X)$ where $X$ is some parameterization of the controller $X = \{x_1, x_2, \ldots, x_n\}$ and $n$ is the total number of parameters. In this paper, both state-space and transfer-function parameterizations of the controller are considered.

1) State Space Parameterizations: Let $(A_g, B_g, C_g, 0)$ be a minimal state-space description of the plant $G(\theta) = C_g(\theta I - A_g)^{-1} B_g$ [which means $G(\theta)$ is strictly proper] $A_g \in \mathbb{R}^{n \times n}$, $B_g \in \mathbb{R}^{n \times 1}$ and $C_g \in \mathbb{R}^{1 \times m}$. Let $(A_h, B_h, C_h, D_h)$ be a state-space description of $H(\theta)$, where $A_h \in \mathbb{R}^{n \times n}$, $B_h \in \mathbb{R}^{n \times q}$, $C_h \in \mathbb{R}^{r \times n}$ and $D_h \in \mathbb{R}^{r \times q}$. In this paper, $(A_h, B_h, C_h, D_h)$ is also called a realization of $H(\theta)$.

Fig. 1. Feedback control system.
The realizations of $H(\theta)$ are not unique. In fact, if $(A^0_h, B^0_h, C^0_h, D^0_h)$ is a realization of $H(\theta)$, then so is $(T^{-1}A^0_hT, T^{-1}B^0_h, C^0_hT, D^0_hT)$ for any nonsingular similarity transformation $T \in \mathbb{R}^{n \times n}$.

For state-space realizations, let $X = \{x_i: i = 1, \ldots, n_x\}$, $n_x = (\ell + n)(q + n)$ be the parameters of the controller

$$X = \begin{bmatrix} D_h & C_h \\ B_h & A_h \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{p+\ell} \\ x_{p+\ell+1} & x_{p+\ell+2} & \cdots & x_2(p+\ell) \\ \vdots & \vdots & & \vdots \\ x_{(\ell+n)(n+\ell)+1} & \cdots & \cdots & x_{(\ell+n)(n+\ell)} \end{bmatrix}.$$ (1)

The transition matrix of the closed-loop system is

$$\mathcal{X} = \begin{bmatrix} A_x + B_x D_h C_h & B_x C_h \\ B_h & A_h \end{bmatrix} = \begin{bmatrix} A_x & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} D_h & C_h \\ B_h & A_h \end{bmatrix} \begin{bmatrix} C_x & 0 \\ 0 & I_n \end{bmatrix}.$$ (2)

Let the realization $(A^0_h, B^0_h, C^0_h, D^0_h)$ of $H(\theta)$ be represented by

$$X_0 = \begin{bmatrix} D_h^0 & C_h^0 \\ B_h^0 & A_h^0 \end{bmatrix}$$ (3)

then, any realization is given by

$$X = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} D_h^0 & C_h^0 \\ B_h^0 & A_h^0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}^{-1}X_0T$$ (4)

where $T \in \mathbb{R}^{p \times n}$ is nonsingular. Hence, the closed-loop matrix $\mathcal{X}(X)$ is given by

$$\mathcal{X}(X) = T^{-1}X_0T.$$ (6)

where $X_0 = \mathcal{X}(X_0)$.

2) Transfer Function Parameterization: Let the SISO plant be

$$G(\theta) = \frac{b(\theta)}{a(\theta)} = \sum_{i=0}^{\alpha_a} \frac{b_i \theta^i}{a_i \theta^i}$$ (7)

where $n_a > n_b$, so $G(\theta)$ is strictly proper. Let the SISO controller be

$$H(\theta) = \frac{q(\theta)}{p(\theta)} = \sum_{i=0}^{\alpha_p} \frac{q_i \theta^i}{p_i \theta^i}$$ (8)

where $n_p > n_q$, so $H(\theta)$ is proper. The closed-loop characteristic equation $f(\theta)$ is, thus,

$$f(\theta) = a(\theta)p(\theta) - b(\theta)q(\theta).$$ (9)

For transfer-function realizations, let $X = \{x_i: i = 1, \ldots, n_x\}$, $n_x = n_p + n_q + 2$, be the parameters of the controller, where $p_i = x_{i+1}$ for $i = 0, \ldots, n_p$, where $n_p$ is the order of the controller denominator and $q_i = x_{i+n_p+2}$ for $i = 0, \ldots, n_q$, where $n_q$ is the order of the controller numerator. The closed-loop characteristic equation is, thus, a function of $X$, and is written as $f(\theta, X)$.

B. A Controller Fragility Measure

A measure of the fragility of the controller is proposed as

$$\Psi = \sum_{k=1}^{N} w_k \Psi_k$$ (10)

where $N$ is the number of closed-loop poles/eigenvalues, $w_k$ is a nonnegative real scalar weighting and

$$\Psi_k = \sum_{i=1}^{n} \left( \partial \lambda_k \right)^2$$ (11)

where $\{\lambda_i: i = 1, \ldots, m+n\}$ represents the set of unique closed-loop poles/eigenvalues and $\{x_i: i = 1, \ldots, n_x\}$ are the controller parameters for either state space or transfer function realizations. The measure is a weighted sum of a 2-norm of the sensitivity of the individual closed-loop system poles/eigenvalues to perturbations in the controller parameters, and is a computable alternative to the norm measures proposed in [15], [16]. A similar measure for the open-loop eigenvalue sensitivities was proposed in [14, p. 139].

1) State Space Parameterization: The following lemma and theorem are required. Lemma 1 is from [15], [16]. Theorem 1 is a well-known result, a proof can be found in [14, p. 136].

**Lemma 1:** Let $f(M) \in C$ be a differentiable function of a matrix $M \in \mathbb{R}^{m \times m}$, and let $M = M_0 + M_1X M_2$, where $M_k \forall k$ are independent of $X$. Denote $F(X) \equiv f(M)$ and

$$\frac{\partial f(M)}{\partial M} \equiv \left\{ \frac{\partial f(M)}{\partial m_{ij}} \right\}.$$ (12)

Then,

$$\frac{\partial F(X)}{\partial X} = M_0^T \frac{\partial f(M)}{\partial M} M_2^T$$ (13)

where “$T$” denotes the transpose operation.

**Theorem 1:** Let $M \in \mathbb{R}^{m \times m}$ have distinct eigenvalues $\{\lambda_i\} = \lambda(M)$. Denote $R = [R_1, R_2, \ldots, R_n]$ as a matrix formed by the right eigenvectors of $M$ corresponding to the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, i.e., $MR_k = \lambda_k R_k$ for $k = 1, \ldots, n$. Let $\Lambda \equiv \text{diag}(\lambda_1, \ldots, \lambda_n)$; hence, $MR = RA$. Since the $\{\lambda_k\}$ are distinct, $R$ is nonsingular, $R^{-1}MR = \Lambda$. Define $L = [L_1, L_2, \ldots, L_n] \equiv R^{-1}$, then, $L^T MR = \Lambda$. Differentiating with respect to $M$ gives the sensitivity of the $k$th eigenvalue to small changes in components of $M$, i.e.,

$$\frac{\partial \lambda_k}{\partial M} = \frac{\partial L_k^T MR_k}{\partial M} = (R_kL_k)^T$$ (14)

where “$T$” denotes the transpose and conjugate operation.

For a state-space parameterization of the controller $X$, then, $\Psi_k$ is

$$\Psi_k = \left\| \frac{\partial \lambda_k}{\partial X} \right\|_F^2$$ (15)

where $\| \cdot \|_F$ denotes the Frobenius norm. Since the $k$th closed-loop eigenvalue $\lambda_k$ is a function of $\mathcal{X}(X) = M_0 + M_1X M_2$ defined by (2), the sensitivity of the $\lambda_k$ with respect to the controller realization matrix $X$ is, from Lemma 1

$$\frac{\partial \lambda_k}{\partial X} = M_1^T \frac{\partial \lambda_k}{\partial A} M_2^T.$$ (16)
Thus, from Theorem 1

\[
\left( \frac{\partial \lambda_k}{\partial X} \right)^T = M_2 R_k L_k^H M_1 \tag{17}
\]

where \( R_k \) and \( L_k \) are the right and left eigenvectors respectively for the \( k \)th eigenvalue of \( A_k \).

Let \( R_k = (R_k^T(1)R_k^T(2))^T \) and \( L_k = (L_k^T(1)L_k^T(2))^T \) be the right and left eigenvectors respectively for the \( k \)th eigenvalue of \( A_k \) partitioned such that \( R_k(1), L_k(1) \in \mathbb{C}^{n_r} \) and \( R_k(2), L_k(2) \in \mathbb{C}^{n_s} \), i.e., the partitions correspond to the partitions of \( A \) defined by (1). From (17) and (1), it can be shown [15] that

\[
\begin{align*}
\left( \frac{\partial \lambda_k}{\partial A_k} \right)^T &= R_k(2)L_k^H(1)B_g + \beta \left( \frac{\partial \lambda_k}{\partial D_k} \right)^T \tag{18}
\end{align*}
\]

Thus, from (15)

\[
\Psi_k = \left| \frac{\partial \lambda_k}{\partial A_k} \right|^2_Y + \left| \frac{\partial \lambda_k}{\partial B_k} \right|^2_Y + \left| \frac{\partial \lambda_k}{\partial C_k} \right|^2_Y + \left| \frac{\partial \lambda_k}{\partial D_k} \right|^2_Y \tag{19}
\]

giving

\[
\Psi_k = \text{tr} \left\{ R_k(2)L_k^H(1)B_g + \beta \left( \frac{\partial \lambda_k}{\partial D_k} \right)^T \right\}^2
\]

which can be rearranged to

\[
\Psi_k = \text{tr} \left\{ R_k^0(2)R_k(2) \right\} \text{tr} \left\{ L_k^H(2)L_k(2) \right\}
\]

then

\[
\Psi_k = \text{tr} \left\{ R_k^0(2)R_k(2) \right\} \text{tr} \left\{ L_k^H(2)L_k(2) \right\}
\]

Let \( R_k^0 = (R_k^0T(1)R_k^0T(2))^T \) and \( L_k^0 = (L_k^0T(1)L_k^0T(2))^T \) be the right and left eigenvectors respectively, for the \( k \)th eigenvalue of \( A_k \). Clearly, \( R_k = T_k^{-1} R_k^0 \) and \( L_k = T_k^T L_k^0 \) are the right and left eigenvectors of \( A_k = T_k^{-1} A_k T_k \) for the same \( k \)th eigenvalue. Thus,

\[
\begin{align*}
R_k(1) &= R_k^0(1) \tag{22}
R_k(2) &= T^{-1} R_k^0(2) \tag{23}
L_k(1) &= L_k^0(1) \tag{24}
L_k(2) &= T^T L_k^0(2) \tag{25}
\end{align*}
\]

and, so \( R_k^0(1) \) and \( L_k^0(1) \) are invariant under the realization transformation. Substituting (22)–(25) into (21) gives

\[
\Psi_k = \text{tr} \left\{ R_k^0(2)P^{-1} R_k^0(2) \right\} \text{tr} \left\{ L_k^0(2)P L_k^0(2) \right\}
\]

where

\[
\begin{align*}
\alpha_k &= \text{tr} \left\{ R_k^{0H}(1)C_g^H C_g R_k^0(1) \right\} \tag{27}
\beta_k &= \text{tr} \left\{ R_k^{0H}(1)B_g R_k^0(1) \right\} \tag{28}
\end{align*}
\]

and \( P = TT^T \). So, from (10)

\[
\Psi(P) = \sum_{k=1}^{n+m} \text{tr} \{ P^{-1} M_{R_k} \} \text{tr} \{ P M_{L_k} \}
\]

\[
+ \text{tr} \{ PW_L \} + \text{tr} \{ P^{-1} W_R \} + c \tag{29}
\]

and

\[
\begin{align*}
M_{R_k} &\triangleq w_k^{1/2} R_k^0(2) R_k^{0H}(2) \tag{30}
M_{L_k} &\triangleq w_k^{1/2} L_k^0(2) L_k^{0H}(2) \tag{31}
W_L &\triangleq L_k^0(2) \text{diag} \{ w_1 \alpha_1, \ldots, w_{n+m} \alpha_{n+m} \} L_k^{0H}(2) \tag{32}
W_R &\triangleq R_k^0(2) \text{diag} \{ w_1 \beta_1, \ldots, w_{n+m} \beta_{n+m} \} R_k^{0H}(2) \tag{33}
\end{align*}
\]

are all Hermitian, and

\[
c = \sum_{k=1}^{n+m} \alpha_k \beta_k \tag{34}
\]

2) Transfer Function Parameterization: Let \( \lambda_k \) be a distinct eigenvalue of the closed-loop system, which is also a root of the characteristic equation satisfying \( f(\lambda_k) = 0 \). Let

\[
f(\theta) = \prod_{k=1}^{N} (\theta - \lambda_k) \equiv f(\theta, X) \tag{35}
\]

where \( N \) is the number of closed-loop poles and \( X \) is the parameterization of the controller as in Section II-B. Recall from Section II-B that

\[
\Psi_k = \sum_{i=1}^{n_s} \left( \frac{\partial \lambda_k}{\partial x_i} \right)^2 \tag{36}
\]

where \( n_s = n_r + n_s + 2 \) and note that the sensitivity of the \( k \)th eigenvalue to a parameter \( x_i \) is given by

\[
\frac{\partial \lambda_k}{\partial x_i} = \left[ \frac{\partial f/\partial x_i}{\partial f/\partial \theta} \right]_{\theta=\lambda_k} \tag{37}
\]

So, from (35) and (9)

\[
\frac{\partial \lambda_k}{\partial x_i} = \frac{a(\lambda_k)x_i^{(m-1)}}{\Gamma(\lambda_k)} \tag{38}
\]

for \( i = 1, \ldots, n_s + 1 \), and

\[
\frac{\partial \lambda_k}{\partial x_i} = -\frac{b(\lambda_k)x_i^{(m-2)}}{\Gamma(\lambda_k)} \tag{39}
\]

for \( i = n_r + 2, \ldots, n_r + n_s + 2 \), where

\[
\Gamma(\lambda_k) = a_n c P_n \prod_{i=1}^{n_s} \lambda_k^{(m-1)} - \lambda_i \lambda_k. \tag{40}
\]

So,

\[
\Psi_k = \Gamma^{-2}(\lambda_k) \left( a_k^2 \lambda_k^2 \sum_{i=0}^{n_r} \lambda_k^{(m-1)} + b_k^2 \lambda_k \sum_{i=0}^{n_s} \lambda_k^{(m-1)} \right). \tag{41}
\]
III. FRAGILITY MINIMIZATION FOR CONTROLLER STATE—SPACE PARAMETERIZATIONS

A. Minimization Problem

The problem of minimizing the closed-loop eigenvalue sensitivity $\Psi(P)$ given by (29) can be stated as

$$\min_{P \rightarrow P > 0 \text{ in } \mathbb{R}^n} \Psi(P). \quad (42)$$

B. Optimal Solution

From the following theorem [14], a solution to (42) exists. The proof of the theorem is in [14, pp. 104–105].

Theorem 2: Let

$$R(P) \triangleq f(P) + tr \{PM \} + tr \{P^{-1}W\} \quad (43)$$

where $P, M, W > 0$, and $f(P)$ is a scalar positive differentiable function of $P$. Then, the minimum of $R(P)$ exists and can be achieved for nonsingular $P$.

The next theorem provides a method of solving the problem (42). The proof follows the method in [14, Th. 4.1 and 5.1].

Theorem 3: The necessary condition for the solution of (42) is given by

$$\frac{\partial \Psi(P)}{\partial P} = 0$$

and this solution is unique.

The solution to $\frac{\partial \Psi(P)}{\partial P} = 0$, and, hence, $P_{opt}$, can be obtained by means of the gradient flow technique of Perkins, Helmk, and Moore [19]. For this technique, a matrix differential equation

$$\dot{P}(t) = - \frac{\partial \Psi(P)}{\partial P} \quad (45)$$

where $\Psi(P(t))$ is a positive scalar, will, starting from a positive definite initial condition $P_0$, converge to the solution $P(\infty) = P_{opt}$, where $P(t) = 0$, and where $\frac{\partial \Psi(P)}{\partial P}$ is given by (44).

From the optimal $P_{opt}$, a corresponding optimal transformation matrix $T$ where $P_{opt} = TT^T$ can be constructed as $T = P_{opt}^{1/2}V$ for any orthogonal matrix $V$ [20].

As discussed in [20], the choice of an appropriate initial condition for solving (45) will improve the convergence of (45). Following [20], an initial condition $P_0$ is chosen which minimizes $tr \{PW_0\} + tr \{P^{-1}W_R\}$. A necessary condition for this is that $PW_0 = W_R$. Thus, an initial condition for solving (45) is chosen as a solution to

$$P_0W_0 = W_R. \quad (46)$$

IV. EXAMPLES

The methodology is illustrated on some of the examples of fragile controllers from [1]. For each example, a weighting vector $w$ required by (10) is defined. The optimal solution is presented and the optimal sensitivity compared with the pole sensitivity of the transfer function parameterization from [1] calculated using (41). Note that, in this paper, positive feedback is assumed.

Example 1 ($H_\infty$-Based Optimum Gain Margin Controller): This example is originally from [21, p. 200]. The plant is

$$G(s) = \frac{s - 1}{s^2 - s^2} \quad (47)$$

and the controller is

$$H(s) = \frac{q_0 + \cdots + q_0}{p_0 s^2 + \cdots + p_0} \quad (48)$$

where

$$q_0 = -38.582, \quad p_0 = -67626$$

The eigenvalues of the closed-loop system matrix are

$$\lambda_{1,2} = -0.46661506790457 \pm 14.22988589425562i$$

$$\lambda_{3,4} = -5.33389432095491 \pm 11.32900815891883i$$

$$\lambda_{5,6} = -0.99949043421189 \pm 0.000005092431333i$$

$$\lambda_7 = -1.00000059578799 \pm 0.000005098889724i.$$

Note that these values differ from those of [1]. This is due to the very large sensitivity of the eigenvalues.

The weighting vector was set to $w = (1, 0, 1 \times 10^{-8}, 0, 0.01, 0, 0.01, 0)$. Since the eigenvalues are all complex conjugate pairs, only one eigenvalue of each pair needs to be considered. The eigenvalues pairs closest to the imaginary axis are weighted most heavily.

With the controller in the transfer function form, the sensitivity is $\Psi = 4.6391 \times 10^{16}$. The optimal value was found to be $\Psi_{opt} = 3.5725 \times 10^9$, over a million-fold improvement.

Example 2 ($H_\infty$ Robust Controller): This example is originally from [21, p. 192]. The plant is

$$G(s) = \frac{s - 1}{s^2 + 0.5s + 0.5} \quad (49)$$

and the controller is

$$H(s) = \frac{124.5s^3 + 364.9s^2 + 360.45s + 120}{s^3 + 227.1s^2 + 440.7s + 220}. \quad (50)$$

The eigenvalues of the closed-loop system matrix are

$$\lambda_1 = -100$$

$$\lambda_2 = -0.1$$

$$\lambda_{3,4} = -1.00001864972139 \pm 0.000032230179160i$$

$$\lambda_5 = -0.99996270055733.$$

Again, these values differ slightly from those of [1]. The weighting vector was set to $w = (1 \times 10^{-6}, 1, 0.1, 0, 0.1, 0.1)$. With the controller in the transfer function form, the sensitivity is $\Psi = 2.6771 \times 10^{27}$. The very-large value is because three of the closed-loop poles are very close to each other. The optimal value was found to be $\Psi_{opt} = 2.8782 \times 10^{15}$.

Example 3 ($H_\infty$ Optimal Control): This example is originally from [22, p. 51, p. 342]. The plant is

$$G(z) = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_3 z^3 + a_2 z^2 + a_1 z + a_0} \quad (51)$$
The weighting vector was set to $w = (1 \times 10^{-6}, 1 \times 10^{-6}, 0.1, 0.1, 0.1, 0.1, 1, 1)$. With the controller in the transfer function form, the sensitivity is $\Psi = 6.7663 \times 10^{14}$. Again, the very large value is because all of the closed-loop poles are very close to at least one other pole. A sensitivity value for a $P_0$ which satisfies (46) was found to be $\Psi(P_0) = 1.185967698 \times 10^{16}$. The value of $c$ defined by (34) is $c = 1.185961024923100 \times 10^{16}$. Clearly, from (29), $\Psi_{opt} > c$, and since $\Psi(P_0)$ is relatively very close to $c$, the initial solution, $P_0$, is near-optimal and is, hence, a suitable solution.

The poles of the closed-loop system are

\[ \lambda_1 = -100.000067560014 \]
\[ \lambda_2 = -99.999324393721 \]
\[ \lambda_{3,4} = -0.49999628602741 \pm 1.322903770158i \]
\[ \lambda_{5,6} = -0.50000371276495 \pm 1.32284754002195i \]
\[ \lambda_7 = -1.00003425338567 \]
\[ \lambda_8 = -0.99996574913740. \]

Again, these values differ slightly from those of [1]. The weighting vector was set to $w = (1 \times 10^{-6}, 1 \times 10^{-6}, 0.1, 0.1, 0.1, 0.1, 1, 1)$.

V. COMMENTS AND CONCLUSION

In this note, a method for reducing the fragility of controllers by a state space parameterization of the controller is presented. The method is based on a weighted norm of the closed-loop pole/eigenvalue sensitivities to controller parameter perturbations. Conditions for the optimal state-space realization of the controller are given and a numerical method for obtaining the solution is presented. The method is demonstrated on some of the examples of [1]. These examples show that the fragility of a controller depends on the realization of the controller.

The sensitivity of the closed-loop poles/eigenvalues is infinite if the closed-loop poles/eigenvalues are not unique. Examples 1, 3 and 6 have closed-loop eigenvalues that are very close to each other; the resulting high sensitivity of the closed-loop poles/eigenvalues means that the calculation of the values of the closed-loop poles/eigenvalues is poorly conditioned and not very accurate. Since the calculation of the sensitivity is dependent on the values of the closed-loop poles/eigenvalues, the actual calculated values of the sensitivity may not be very reliable for Examples 1, 3 and 6. However, the calculated sensitivities are enormously reduced by the proposed method. As discussed in [2], the examples of [1] are not realistic control designs, and are in a sense contrived, and in most real controller designs it is unlikely that the closed-loop eigenvalues would be assigned so close to each other. Hence, the proposed method is suitable for real applications where there is a fragility problem. As pointed out in [2], the examples also illustrate the importance of posing the design problem correctly to consider the controller uncertainty. In [23], a method for this has been presented.

In this note, only state-space controller parameterization solutions were considered. The pole/zero parameterization was briefly discussed by [2]. Other possible parameterizations include lattice filters [7, pp. 307–310] and the descriptor state-space structure [24].
The method presented in this note considers only the closed-loop stability. To consider the performance, the sensitivity of the closed-loop eigenvectors needs to be included in the fragility measure.

A number of other methodologies could also be considered for the fragility problem. A small-gain technique has been proposed by [25], however, that approach is fairly conservative. Clearly, the parameters of an implemented controller are real, and the uncertainty on these parameters will be real and time invariant. Techniques based on the real stability radius [26] would take this into account, work has been done by [9]. The parametric approach to uncertainty, for example [27] and [28], is clearly another alternative approach.

In addition, as pointed out in [2], there has been a lot of work done on controller order reduction, this is another area closely related to the fragility problem. Guaranteed bounds on the system performance subject to controller uncertainty resulting from the order reduction have been obtained by, for example, [29].

REFERENCES


Sequential Identification and Control for Bounded-Noise ARX Systems

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Abstract—An optimal combination of sequential identification and control for a linear bounded-input bounded-noise discrete-time ARX plant is considered. Various configurations of identifying/controlling sequences are investigated in order to find an optimal tradeoff. The optimal identifying input sequence is determined by maximization of a given identification accuracy measure while the control input sequence is derived to assure a good tracking of the control system. A second-order model is taken for simulation and optimization.

Index Terms—Bounded ARX system, control, identification, optimal tradeoff.

I. INTRODUCTION

The sequential identification and control can be considered as an alternative approach to the simultaneous identification and control, i.e., self-tuning. The reasons to use this approach could be the problems with convergence of recursive parameter estimator which occur in adaptive closed-loop or poor performance of adaptive control, see [1]